Field algebra, Hilbert space and observables in two-dimensional higher-derivative field theories

2002 J. Phys. A: Math. Gen. 35 6033

(http://iopscience.iop.org/0305-4470/35/29/307)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **35** (2002) 6033–6048 PII: S0305-4470(02)34529-3

Field algebra, Hilbert space and observables in two-dimensional higher-derivative field theories

L V Belvedere and R L P G do Amaral

Instituto de Física—Universidade Federal Fluminense, Av. Litorânea, S/N, Boa Viagem, Niterói, CEP 24210-340, Rio de Janeiro, Brazil

E-mail: belve@if.uff.br and rubens@if.uff.br

Received 1 March 2002
Published 12 July 2002
Online at stacks.iop.org/JPhysA/35/6033

Abstract

We discuss structural aspects related to the field algebra of two-dimensional higher-derivative quantum field theories. We present general selection criteria for the proper field subalgebra that generates Wightman functions satisfying the asymptotic factorization property and which define a semi-definite inner product Hilbert space. The positive definite inner product Hilbert space, which contains as a subspace of states the general Wightman functions of the corresponding standard canonical models, is a quotient space obtained by equivalence classes. For higher-derivative local gauge theories, besides the Lowenstein–Swieca condition, an additional condition must be imposed on the field algebra in order to obtain a physical subspace of states satisfying a reasonable set of physically meaningful axioms.

PACS numbers: 11.10.Ef, 11.10.Kk, 11.15.Tk

1. Introduction

In the last few years, an impressive effort has been made by many physicists to understand the underlying properties of quantum field theories in two dimensions [1], as well as to try to picture these models as theoretical laboratories to obtain an insight into more realistic four-dimensional field theories and, more recently, to apply them to low-dimensional condensed matter systems [2], as well as to N-body problems in nuclear physics [3]. After over a quarter of a century of investigations on two-dimensional field theories we have learnt that, besides their peculiar formal aspects, two-dimensional models have also value for providing a better conceptual and structural understanding of general properties of quantum field theory [4–6].

As pointed out in [7–10], the use of bosonization techniques [11] to study two-dimensional quantum field theories raises some non-trivial and delicate questions related to the use of a redundant Bose field algebra. The Bose field algebra contains more degrees of freedom than those needed for the description of the model and some care must be taken in order to construct the Hilbert space associated with the Wightman functions that define the model. The lack of sufficient appreciation of the general mathematical structures involved may give rise to misleading conclusions about the basic structural aspects and physical properties of the models.

Two-dimensional higher-derivative field theories have been discussed in [12–15] and analysed by functional methods and operator formalism. Similar to standard two-dimensional quantum electrodynamics (Schwinger model) [17, 19], the local gauge formulation of the higher-derivative model exhibits non-perturbative features such as anomalous axial divergence and mass generation for the gauge field. The use of higher-derivative Lagrangian models has been the focus of great attention recently in the context of non-Abelian field quantization [20, 21].

The purpose of the present paper is to analyse some mathematical and structural aspects of higher-derivative quantum field theories, which have not been fully appreciated and clarified in the preceding literature. This streamlines the presentations of [12–15]. It is the aim of this paper to show, in a precise manner, what is the effect of the higher-derivatives in the physical content of the model. According to the general principles of local quantum field theory, we shall adopt the strategy of performing the construction of the Hilbert space of the model based on the intrinsic local field algebra defining the theory in order to give a consistent explanation of the mathematical structure and the observable content of the two-dimensional higher-derivative quantum field theories.

This paper is organized as follows. In section 2 we resume the essential features of the operator solution of the higher-derivative free massless theory. We analyse the free field algebra and the Hilbert space hierarchical structure of the higher-derivative free theory. We show that the space of states containing the general Wightman functions of the canonical free Fermi theory can be constructed from a proper field subalgebra which is represented in a positive semidefinite inner product Hilbert subspace. In section 3 we resume the operator solution of the higher-derivative gauge theory and make some remarks on the local field algebra of observables and the corresponding Hilbert space. We construct a gauge invariant bilocal operator for the higher-derivative theory, which is the generalization of the bilocal operator introduced by Lowenstein and Swieca for standard QED_2 [17]. The gauge invariant current is obtained as the point-splitting limit procedure of the bilocal operator. It has been conjectured [12] that in a higher-derivative theory the electric charge is more confined than in the corresponding standard model. In analogy with what happens in the massless Schwinger model, the total charge at the end of strings is unstable and oscillates periodically in time no matter how small is the separation between the charges.

2. Free theory

In order to make this paper self-contained, in this section we present a brief review of the operator solution of the free massless theory with higher-derivatives[1]. In section 2.1, within the framework of general principles of quantum field theory we discuss some mathematical and structural aspects of the free field algebra and the Hilbert space hierarchy.

The higher-derivative free massless theory of order $\mathcal{N}$ is defined by the Lagrangian density [12–16],

$$\mathcal{L}_o = \mathrm{i}\bar{\xi}\, \partial\!\!\!/^{(\mathcal{N})} \xi \tag{2.1}$$

[1] For more details see [13–15].

where the odd- and even-derivatives are defined by

$$\not{d}^{(2N+1)} \doteq [\not\partial\not\partial^{\dagger}]^{N}\not\partial \qquad N = 0, 1, \ldots \tag{2.2}$$

$$\not{d}^{(2N)} \doteq \gamma^{0}[\not\partial\not\partial^{\dagger}]^{N} \qquad N = 1, 2, \ldots . \tag{2.3}$$

The canonical quantization of the free theory with higher-derivative of order $\mathcal{N}$ is performed considering the configuration space generated by the field $\xi_{(\alpha)}(x)$ ($\alpha = 1, 2$, are spinor indices) and its $(\mathcal{N} - 1)$ space–time derivatives[2],

$$\xi^{n}_{(\alpha)} \doteq (\partial_{\mp})^{n}\xi_{(\alpha)} \qquad n = 0, 1, \ldots, \mathcal{N} - 1. \tag{2.4}$$

The corresponding associated momenta are given by

$$\pi^{n}_{(\alpha)} = c_{n}(\partial_{\mp})^{(\mathcal{N}-1-n)}\xi^{*}_{(\alpha)} \tag{2.5}$$

where the c_n are constants [13, 15]. The canonical quantization using the Dirac method leads to the equal-time anticommutation relations

$$\left\{\xi^{n}_{(\alpha)}(x), \pi^{m}_{(\alpha)}(y)\right\}_{x^0=y^0} = \mathrm{i}\delta_{nm}\delta(x^1 - y^1). \tag{2.6}$$

The general equal-time anticommutation relations are given in terms of the derivatives of the field $\xi_{(\alpha)}(x)$ by

$$\left\{\partial^{p}_{\mp}\xi_{(\alpha)}(x), \partial^{q}_{\mp}\xi^{*}_{(\alpha)}(y)\right\}_{x^0=y^0} = \mathrm{i}(-1)^{N-p-1}\delta_{p+q,\mathcal{N}-1}\delta(x^1 - y^1). \tag{2.7}$$

The operator solution $\xi_{(\alpha)}(x)$ for the free massless theory is given in terms of $\mathcal{N}$ free and massless Dirac fields (infrafermions) $\psi^{j}_{(\alpha)}(x^{\pm})$ by the expansion [14, 15]

$$\xi_{(\alpha)}(x) = \sum_{j=1}^{\mathcal{N}} f_j(x^{\mp})\psi^{j}_{(\alpha)}(x^{\pm}) \tag{2.8}$$

where $f_j(x^{\pm})$ are space–time functions[3] [13–15]. For $\mathcal{N} = 2N$ ($\mathcal{N} = 2N + 1$), the Dirac fields $\psi^{j}_{(\alpha)}(x^{\pm})$ are quantized with positive (negative) metric for $j \leqslant N$ ($j > N$).

The $2n$-point correlation functions of the field $\xi(x)$ violate the cluster decomposition property. For instance, the two-point function is given by

$$\langle 0|\xi^{*}_{(\alpha)}(x)\xi_{(\alpha)}(y)|0\rangle \propto (x^{\mp} - y^{\mp})^{\mathcal{N}-1}S_{\alpha\alpha}(x^{\pm} - y^{\pm}) \qquad \alpha = 1, 2 \tag{2.9}$$

where $S_{\alpha\alpha}(z^{\pm})$ is the two-point function of the free and massless canonical Fermi field,

$$S_{\alpha\alpha}(z^{\pm}) = \frac{1}{2\pi\mathrm{i}}\frac{1}{z^{\pm} + \mathrm{i}\epsilon} \qquad \alpha = 1, 2. \tag{2.10}$$

In (2.9), the factorized increasing and unbounded contribution that violates the asymptotic factorization property arises from the space–time functions $f_j(x^{\pm})$.

The bosonized form of the free Fermi fields is given by

$$\psi^{j}_{(\alpha)}(x) = \left(\frac{\mu}{2\pi}\right)^{1/2}\mathcal{K}_{\alpha}\mathcal{K}_{j} : \mathrm{e}^{\mathrm{i}\sqrt{\pi}[\gamma^{5}_{\alpha\alpha}\tilde{\phi}_j(x)+\phi_j(x)]} : \tag{2.11}$$

where $\mathcal{K}_j$ and $\mathcal{K}_\alpha$ are, respectively, the Klein factors needed to ensure the anticommutation relations for the different ψ^j fields and different spinor field components $\psi_{(\alpha)}$ [11, 13, 16], and ϕ_j ($\tilde{\phi}_j$) are $\mathcal{N}$ independent massless scalar (pseudo-scalar) fields. Similar to what is done in [13, 14], we introduce the following decomposition for the $\mathcal{N}$ independent scalar fields:

$$\phi_j = \frac{1}{\sqrt{\mathcal{N}}}\Phi + \sum_{i_D=1}^{\mathcal{N}-1}\lambda^{i_D}_{jj}\varphi^{i_D} \tag{2.12}$$

[2] We follow the notation and conventions used in [13–15], $x^{\pm} = x^0 \pm x^1$, $\partial_{\pm} = \partial_0 \pm \partial^1$.

[3] See appendix A.

where $\lambda^{i_D}_{jj}$ are the diagonal matrices of the $SU(\mathcal{N})$ group. The Φ field acts as potential for $U(1)$ and chiral-$U(1)$ conserved free currents, J^{μ}_{f} and $J^{\mu^5}_{f}$, respectively. The corresponding dual field $\tilde{\Phi}$ is defined by

$$\partial_\mu \tilde{\Phi} + \epsilon_{\mu\nu}\partial^\nu \Phi = 0 \tag{2.13}$$

and obeys a local Gauss's law [7]

$$J^{\mu}_{\text{free}} = \sqrt{\frac{\mathcal{N}}{\pi}}\partial\tilde{\Phi} = \partial^\nu F_{\nu\mu} \tag{2.14}$$

with $F_{\nu\mu} = \sqrt{\mathcal{N}/\pi}\,\epsilon_{\nu\mu}\Phi$.

Using decomposition (2.12) in the bosonized expression (2.11), the $U(1)$ dependence factorizes and we can write

$$\xi_{(\alpha)}(x) = S^{\Phi}_{(\alpha)}(x^{\pm})\sum_{j=1}^{\mathcal{N}} f_j(x_{\mp})\hat{\psi}^{j}_{(\alpha)}(x^{\pm}) \tag{2.15}$$

where the 'soliton' operator S^{Φ} and the $U(1)$-screened Dirac infrafermions $\hat{\psi}^j$ are given by [13, 15, 18]

$$S^{\Phi}_{(\alpha)}(x^{\pm}) =: \exp\left\{2\mathrm{i}\sqrt{\frac{\pi}{\mathcal{N}}}\Phi(x^{\pm})\right\}: \tag{2.16}$$

$$\hat{\psi}^{j}_{(\alpha)}(x^{\pm}) = \left(\frac{\mu}{2\pi}\right)^{\frac{1}{2}} : \exp\left\{2\mathrm{i}\sqrt{\pi}\sum_{i_D=1}^{\mathcal{N}-1}\lambda^{i_D}_{jj}\varphi^{i_D}(x^{\pm})\right\} : . \tag{2.17}$$

This summarizes the general properties of the operator solution of the free theory which are relevant for our present purpose.

2.1. Free field algebra and Hilbert space

At first glance the field $\xi(x)$ is a defiled operator-valued field given in terms of a superposition of $\mathcal{N}$ massless free Fermi field operators $\psi^j(x^{\mp})$, weighted by the increasing space–time functions $f_j(x^{\pm})$, which in turn contribute to violate the asymptotic factorization property. Nevertheless, the field algebra generated by $\xi(x)$ and its derivatives satisfies the principle of Einstein causality and is a local field algebra. The infrafermion fields ensure, for instance, the correct microcausality requirements [16], and in this sense, the main quantum field features of the operator solution (2.8) are implemented by the infrafermion field operators.

In order to construct from the Lagrangian density (2.1) the corresponding quantum field theory, we shall adopt as a basic assumption that in a higher-derivative field theory of order $\mathcal{N}$, the field algebra intrinsic to the model is enlarged by the inclusion of the $\mathcal{N}-1$ derivatives of the field $\xi(x)$. In this way, the net of the algebraic structure defining the model is generated by the set of field operators $\{\xi^0,\ldots,\xi^{\mathcal{N}-1},\pi^0,\ldots,\pi^{\mathcal{N}-1}\}$. Within this approach, this set of field operators constitutes the intrinsic mathematical description of the theory and serves as a kind of building material whose Wightman functions define the theory. The higher-derivative free theory is then defined by the local field algebra $\boldsymbol{A}$ generated by the polynomials of the smoothed local fields $\{\xi^n(h),\pi^n(h)\}$, $n = 0, 1, \ldots, \mathcal{N}-1$, satisfying the locality condition

$$\{\mathcal{O}(f),\mathcal{O}(h)\} = 0 \qquad \mathcal{O}\in\boldsymbol{A} \tag{2.18}$$

if f is spacelike relative to h. The field algebra $\boldsymbol{A}$ is represented in the indefinite inner product Hilbert space $H \equiv H_{\{\xi^0,\ldots,\xi^{\mathcal{N}-1},\pi^0,\ldots,\pi^{\mathcal{N}-1}\}}$,

$$H \doteq \boldsymbol{A}|0\rangle. \tag{2.19}$$

In general, the correlation functions of the fields ξ^k and π^ℓ violate the asymptotic factorization property. The space–time clustered contributions of the light-cone variable $x^\pm$, arising from the infrafermion correlation functions, are weighted by an increasing space–time contribution of the light-cone variable $x^\mp$, arising from the functions $f_j(x^\mp)$. Nevertheless, the general Wightman functions of the canonical free Dirac field $\psi^o_{(\alpha)}$, satisfying the positive-definiteness condition, can be obtained considering correlations between the configuration space variable $\xi^m(x)$ and the corresponding associated momenta $\pi^m(x)$ [15]. For instance, for the $2p$-point function for fixed m we obtain the isomorphism

$$\langle 0| \prod_{j=1}^{p} \pi^m_{(\alpha)}(x_j) \prod_{k=1}^{p} \xi^m_{(\alpha)}(y_k)|0\rangle \equiv Z\langle 0| \prod_{j=1}^{p} \psi^{o^*}_{(\alpha)}(x_j^\pm) \prod_{k=1}^{p} \psi^o_{(\alpha)}(y_k^\pm)|0\rangle \tag{2.20}$$

where Z is a finite constant.

It is worth remarking that in higher-derivative free field theory of odd order $\mathcal{N} = 2N + 1$, the subspace of states containing the general Wightman functions of the canonical free Fermi theory can be constructed from a proper field subalgebra $\widetilde{\boldsymbol{A}} \subset \boldsymbol{A}$. The field subalgebra $\widetilde{\boldsymbol{A}}$ is represented in a positive semidefinite inner product Hilbert subspace $\widetilde{H} \subset H$,

$$\widetilde{H} \doteq \widetilde{\boldsymbol{A}}|0\rangle \tag{2.21}$$

in which the cluster decomposition property is satisfied. As we shall see, the field subalgebra $\widetilde{\boldsymbol{A}}$ can be defined according to a subsidiary algebraic condition. In order to illustrate this point, consider for instance the particular case $\mathcal{N} = 3$. According to equation (2.8), the configuration space variables that are elements of the intrinsic field algebra $\boldsymbol{A}$ can be written as

$$\xi^0(x) = \xi(x) = [\pi^2(x)]^* = a\chi_+(x^\pm) + (x^\mp)^2 b\chi_-(x^\pm) + x^\mp\psi^2(x^\pm) \tag{2.22}$$

$$\xi^1(x) = \partial_\mp\xi(x) = [\pi^1(x)]^* = 2x^\mp b\chi_-(x^\pm) + \psi^2(x^\pm) \tag{2.23}$$

$$\xi^2(x) = \partial^2_\mp\xi(x) = [\pi^0(x)]^* = 2b\chi_-(x^\pm) \tag{2.24}$$

where a and b are constants, the field ψ^2 is a canonical free Fermi field, quantized with positive metric, the fields ψ^1 and ψ^3 are quantized with opposite metric. The fields $\chi_\pm$ are defined by

$$\chi_\pm(x^\pm) \doteq \psi^1(x^\pm) \pm \psi^3(x^\pm) \tag{2.25}$$

and generate zero-norm states from the vacuum,

$$\||\chi_\pm\rangle\|^2 = \langle 0|\chi_\pm(x)\chi_\pm(y)|0\rangle = 0. \tag{2.26}$$

In order to make clear the identification of the elements of the intrinsic field algebra, we shall write the basic fields $\{\xi^0, \xi^1, \xi^2\}$ in a more convenient way. To this end, we define the 'defiled dressed operators'

$$\check{\chi}_+(x) \doteq a\chi_+(x^\pm) \tag{2.27}$$

$$\check{\chi}_-(x) \doteq (x^\mp)^2 b\chi_-(x^\pm) \tag{2.28}$$

$$\check{\psi}^2(x) \doteq x^\mp\psi^2(x^\pm) \tag{2.29}$$

such that we can write (2.22)–(2.24) as

$$\xi^0(x) = \check{\chi}_+(x^\pm) + \check{\chi}_-(x) + \check{\psi}^2(x) \tag{2.30}$$

$$\xi^1(x) = \partial_\mp[\check{\chi}_-(x) + \check{\psi}^2(x)] \tag{2.31}$$

$$\xi^2(x) = \partial^2_\mp\check{\chi}_-(x). \tag{2.32}$$

Since by assumption $\xi^2 \in \boldsymbol{A}$, then by (2.32), $\partial^2_\mp \check{\chi}_-(x) = 2b\chi_-(x^\pm) \in \boldsymbol{A}$, implying that $\chi_-(x^\pm) \in \boldsymbol{A}$, and thus the zero-norm field χ_- can be defined as an operator in H.

In order to obtain the subspace of states satisfying a reasonable set of physically meaningful axioms, we define the field subalgebra $\widetilde{\boldsymbol{A}} \subset \boldsymbol{A}$ by the algebraic conditions in H (with $j = 0, 1$, and $N = 1$)

$$\{\xi^i(x), \chi_-^*(y)\} = 0 \qquad \forall (x, y) \tag{2.33}$$

$$\{\pi^j(x), \chi_-(y)\} = 0 \qquad \forall (x, y). \tag{2.34}$$

This restricts i to $i = 1$ or 2 and j to $j = 0$ or 1. The fields satisfying the anticommutation relations above define the field subalgebra $\widetilde{\boldsymbol{A}}$, which is represented in the positive semidefinite inner product Hilbert subspace $\widetilde{H} \doteq \widetilde{\boldsymbol{A}}|0\rangle$, in which the asymptotic factorization property holds.

For the general case $\mathcal{N} = 2N + 1$, we can define N fields χ_-^κ and N fields χ_+^κ, each of them written as a combination of two independent infrafermion fields quantized with opposite metric, such that

$$\| |\chi_\pm^\kappa\rangle \|^2 = 0 \qquad \kappa = 1, \ldots, N. \tag{2.35}$$

The field operator $\xi(x)$ can be written as

$$\xi_{(\alpha)}(x) = \sum_{\kappa=1}^{N} \left[\check{\chi}^\kappa_{+(\alpha)}(x) + \check{\chi}^\kappa_{-(\alpha)}(x)\right] + \check{\psi}^{N+1}_{(\alpha)}(x) \tag{2.36}$$

with

$$\check{\psi}^{N+1}_{(\alpha)}(x) = \frac{(x^\mp)^N}{2^N N!} \psi^{N+1}(x^\pm) \tag{2.37}$$

$$\check{\chi}^\kappa_+(x) = \frac{1}{(\kappa - 1)!\sqrt{2}m^N} \left(\frac{mx^\mp}{2}\right)^{\kappa-1} (\psi^\kappa(x^\pm) + \psi^{2N+2-\kappa}(x^\pm)) \tag{2.38}$$

$$\check{\chi}^\kappa_-(x) = \frac{1}{(2N + 2 - \kappa)!\sqrt{2}m^N} \left(-\frac{mx^\mp}{2}\right)^{2N+2-\kappa} (\psi^\kappa(x^\pm) - \psi^{2N+2-\kappa}(x^\pm)). \tag{2.39}$$

In particular, the field operator ξ^N can be written as

$$\xi^N_{(\alpha)}(x) = \partial^N_\mp \xi_{(\alpha)}(x) = \sum_{\kappa=1}^{N} \partial^N_\mp \check{\chi}^\kappa_{-(\alpha)}(x) + 2^{-N}\psi^{N+1}_{(\alpha)}(x^\pm) \tag{2.40}$$

and

$$\xi^{N+j}_{(\alpha)}(x) = \sum_{\kappa=1}^{N-j} \partial^{N+j}_\mp \check{\chi}^\kappa_{-(\alpha)}(x) \tag{2.41}$$

where ψ^{N+1} is a canonical free and massless Fermi field quantized with positive metric and all other fields appearing are zero-norm fields. We define the field subalgebra $\tilde{\mathcal{A}}$ to ensure the cluster decomposition property through the algebraic conditions in H

$$\left\{\xi^i_{(\alpha)}(x), \left[\sum_{\kappa=1}^{N-l} (\partial_\mp)^{N+l} \check{\chi}^\kappa_-(y)\right]^*\right\} = 0 \qquad \forall (x, y) \quad l = 1, \ldots, N \tag{2.42}$$

$$\left\{\pi^j_{(\alpha)}(x), \sum_{\kappa=1}^{N-l} (\partial_\mp)^{N+l} \check{\chi}^\kappa_-(y)\right\} = 0 \qquad \forall (x, y) \quad l = 1, \ldots, N. \tag{2.43}$$

The subset of intrinsic field operators from which we can construct the field subalgebra $\widetilde{\boldsymbol{A}}$, and which generates Wightman functions satisfying the asymptotic factorization property, is selected according to the following criteria:

Let $\Xi(x)$ be an element of the intrinsic set of field operators which defines the field algebra $\boldsymbol{A}$, and let $\widetilde{\boldsymbol{A}} \subset \boldsymbol{A}$, be the local field subalgebra satisfying the asymptotic factorization property. Then, $\Xi(x) \in \widetilde{\boldsymbol{A}}$ iff the following algebraic constraints are satisfied:

$$\begin{aligned} \{\Xi^*(x), \xi^\ell(y)\} &= 0 \qquad \forall(x,y) \quad \ell = N+1, \ldots, 2N \\ \{\Xi\ (x), \pi^\jmath(y)\} &= 0 \qquad \forall(x,y) \quad \jmath = 0, 1, \ldots, N-1. \end{aligned} \tag{2.44}$$

The field subalgebra $\widetilde{\boldsymbol{A}}$ is represented in the positive semidefinite inner product Hilbert subspace $\widetilde{H}$:

$$\widetilde{H}_{\{\xi^N,\ldots,\xi^{2N},\pi^0,\ldots,\pi^N\}} \subset H_{\{\xi^0,\ldots,\xi^N,\ldots,\xi^{2N},\pi^0,\ldots,\pi^N,\ldots,\pi^{2N}\}} \tag{2.45}$$

with

$$\widetilde{H} \doteq \widetilde{\boldsymbol{A}}|0\rangle. \tag{2.46}$$

Moreover, we get

$$\partial_\mp \xi^N_{(\alpha)}(x) = \sum_{\kappa=1}^{N} \partial_\mp^{N+1} \check{\chi}^\kappa_-(x) \tag{2.47}$$

and conditions (2.44) imply that the configuration space variable $\xi^N_{(\alpha)} = \partial^N_\mp \xi_{(\alpha)}$, as well as the associated momenta π^N, satisfy in the subspace $\widetilde{H}$ the weak form of the first-order free field equations of motion,

$$\partial_\mp \xi^N_{(\alpha)}(x) \approx 0 \qquad \partial_\mp \pi^N_{(\alpha)}(x) \approx 0 \tag{2.48}$$

which means that

$$\langle \Xi|\partial_\mp \xi^N_{(\alpha)}(x)|\Xi\rangle = 0 \qquad \forall|\Xi\rangle \in \widetilde{H} \tag{2.49}$$

$$\langle \Xi|\partial_\mp \pi^N_{(\alpha)}(x)|\Xi\rangle = 0 \qquad \forall|\Xi\rangle \in \widetilde{H}. \tag{2.50}$$

The Hilbert subspace $\widetilde{H}$ contains besides zero-norm states, such as

$$|\pi^o\rangle, \ldots, |\pi^{N-1}\rangle, |\xi^{N+1}\rangle, \ldots, |\xi^{2N}\rangle \tag{2.51}$$

the general Wightman functions of the canonical free Fermi theory, which are generated by the field operators $\xi^N(x)$ and $\pi^N(x)$.

Let us denote by $\widetilde{\boldsymbol{A}}_o \subset \widetilde{\boldsymbol{A}}$ the field subalgebra which generates zero-norm states and $\widetilde{\boldsymbol{A}}_o|0\rangle$ the collection of all zero-norm states of $\widetilde{H}$, such that $\widetilde{H}_o \doteq \widetilde{\boldsymbol{A}}_o|0\rangle \subset \widetilde{H}$ is the zero-norm subspace of $\widetilde{H}$. The states of the Hilbert subspace $\widetilde{H}$ can be accommodated as equivalence classes of $|\xi^N\rangle$, and $|\pi^N\rangle$, modulo $|\widetilde{\boldsymbol{A}}_o\rangle$, so that by successive quotients one gets the positive definite inner product Hilbert space of states of the free Fermi field ψ^{N+1},

$$\widetilde{H}_{\psi^{N+1}} \doteq \frac{\widetilde{H}}{\widetilde{H}_o}. \tag{2.52}$$

The quotient space $\widetilde{H}_{\psi^{N+1}}$ is given by

$$\widetilde{H}_{\psi^{N+1}} \equiv \widetilde{\boldsymbol{A}}_{N+1}|0\rangle \tag{2.53}$$

where the field subalgebra $\widetilde{\boldsymbol{A}}_{N+1} \subset \widetilde{\boldsymbol{A}}$ is generated by $\tilde{\pi}^N$ and $\tilde{\xi}^N = (\tilde{\pi}^N)^* \equiv \psi^{N+1}$. The quotient space $\widetilde{H}_{\psi^{N+1}}$ is isomorphic to the Hilbert space of the two-dimensional free massless

Fermi theory. In the space $\widetilde{H}_{\psi^{N+1}}$ the first-order free field equation of motion is satisfied in the strong form

$$\partial_{\mp}\xi^N_{(\alpha)}(x) = 0. \tag{2.54}$$

It must be stressed that the higher-derivative free theory of odd order cannot be considered as being physically equivalent to the free canonical Fermi theory. The space of states H contains, as a proper subspace, the Hilbert space $\widetilde{H}_{\psi^{N+1}}$, which provides a representation of the free field algebra generated by the field operator ψ^{N+1}.

We now consider the algebraic structure of theories with higher derivatives of even order. In this case we can also define N fields χ^k_- and N fields χ^k_+, each of them written as a combination of two independent infrafermion fields quantized with opposite metric, such that

$$\langle \chi^\kappa_\pm(x)\chi^\kappa_\pm(y)\rangle = 0 \qquad \kappa = 1, \ldots, N. \tag{2.55}$$

Consider for instance the case $\mathcal{N} = 2$. The configuration space variables that are elements of the intrinsic field algebra $\boldsymbol{A}$ can be written as

$$\xi^0(x) = a\chi_+(x^\pm) + \check{\chi}_-(x) \tag{2.56}$$

$$\xi^1(x) = \partial_\mp\check{\chi}_-(x) \tag{2.57}$$

where

$$\check{\chi}_-(x) \doteq bx^\mp\chi_-(x^\pm) \tag{2.58}$$

$$\chi_\pm \doteq \psi^1(x^\pm) \pm \psi^2(x^\pm) \tag{2.59}$$

and the Fermi fields ψ^1 and ψ^2 are quantized with opposite metric. The field operator ξ can be written as

$$\xi_{(\alpha)}(x) = \sum_{\kappa=1}^{N}\left[\check{\chi}^\kappa_{+(\alpha)}(x) + \check{\chi}^\kappa_{-(\alpha)}(x)\right]. \tag{2.60}$$

Following the previous procedure, we obtain that the field subalgebra $\widetilde{\boldsymbol{A}}$ is represented in a zero-norm subspace

$$\widetilde{H}^{\text{even}}_o \doteq \widetilde{\boldsymbol{A}}|0\rangle \subset H^{\text{even}}. \tag{2.61}$$

In this case, we get the following weak conditions

$$\xi^N(x) \approx 0 \qquad \pi^{N-1}(x) \approx 0 \tag{2.62}$$

which means that

$$\langle\Xi|\xi^N(x)|\Xi\rangle = 0 \qquad \forall|\Xi\rangle \in \widetilde{H}^{\text{even}}_o \tag{2.63}$$

$$\langle\Xi|\pi^{N-1}(x)|\Xi\rangle = 0 \qquad \forall|\Xi\rangle \in \widetilde{H}^{\text{even}}_o. \tag{2.64}$$

Since in this case we cannot isolate the field subalgebra of a canonical-free Fermi field, the Hilbert space of the even-order theory is completely devoid of any physical content.

3. Local gauge theory

The two-dimensional quantum electrodynamics with higher derivatives of order $\mathcal{N}$ is defined by the Lagrangian density [12–15]

$$\mathcal{L} = -\tfrac{1}{4}(\mathcal{F}_{\mu\nu})^2 + \mathrm{i}\bar{\Psi}\not{\Delta}^{(\mathcal{N})}\Psi \tag{3.1}$$

where the covariant derivatives of *odd*- and *even*-order are defined by

$$\not{\Delta}^{(2N+1)} \doteq [\not{\mathcal{D}}\not{\mathcal{D}}^{\dagger}]^{N}\not{\mathcal{D}} \qquad N = 0, 1, \ldots \tag{3.2}$$

$$\not{\Delta}^{(2N)} \doteq \gamma^{0}[\not{\mathcal{D}}\not{\mathcal{D}}^{\dagger}]^{N} \qquad N = 1, 2, \ldots \tag{3.3}$$

respectively. The usual covariant derivative is given by $\not{\mathcal{D}} = \gamma^{\mu}(\partial_{\mu} - \mathrm{i}e\mathcal{A}_{\mu})$.

In a local gauge formulation, the higher-derivative theory is defined by the equations of motion

$$\not{\Delta}^{(\mathcal{N})}\Psi(x) = 0 \tag{3.4}$$

$$\partial^{\nu}\mathcal{F}_{\mu\nu}(x) = -e(\mathrm{J}_{\mu}(x) + \ell_{\mu}(x)). \tag{3.5}$$

In Gauss's law (3.5), ℓ_{μ} is a longitudinal current with no observable content and the covariant derivative of the field operator $\Psi(x)$ in (3.4) is defined by the point-splitting limit procedure,

$$\not{\mathcal{D}}\Psi(x) \doteq \gamma^{\mu}\partial_{\mu}\Psi(x) - \frac{1}{2}\mathrm{i}e \lim_{\substack{\epsilon\to 0\\ \epsilon^{2}<0}} \gamma^{\mu}\{\mathcal{A}_{\mu}(x+\epsilon)\Psi(x) + \Psi(x)\mathcal{A}_{\mu}(x-\epsilon)\}. \tag{3.6}$$

As in the standard Schwinger model [17, 18], the decoupling of the gauge field is obtained by performing a chiral q-number gauge transformation on the free field operator. The operator solution in the Landau gauge is given by

$$\Psi_{(\alpha)}(x) =: \mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{\mathcal{N}}}\gamma^{5}_{\alpha\alpha}[\Sigma(x)+\tilde{\eta}(x)]} : \xi_{(\alpha)}(x) \tag{3.7}$$

where the free massless pseudo-scalar field $\tilde{\eta}$ is quantized with negative metric and Σ is a massive scalar field[4],

$$\left[\Box + \frac{\mathcal{N}e^{2}}{\pi}\right]\Sigma(x) = 0. \tag{3.8}$$

The gauge field is identified as being given by

$$\mathcal{A}_{\mu}(x) = -\frac{1}{e}\sqrt{\frac{\pi}{\mathcal{N}}}\varepsilon_{\mu\nu}\partial^{\nu}(\Sigma(x) + \tilde{\eta}(x)). \tag{3.9}$$

The phase space of the higher-derivative gauge theory is generated by

$$\Psi^{n}_{(\alpha)} = \mathcal{D}^{n}_{\mp}\Psi_{(\alpha)} \tag{3.10}$$

and the associated momenta,

$$\Pi^{n}_{(\alpha)} = c_{n}\left(\mathcal{D}^{(\mathcal{N}-1-n)}_{\mp}\Psi_{(\alpha)}\right)^{*} \tag{3.11}$$

which satisfy canonical anticommutation relations. The phase space variables $\{\Psi^{n}, \Pi^{n}\}$ can be written in terms of the free theory variables $\{\xi^{n}, \pi^{n}\}$ as [13–15]

$$\Psi^{n}_{(\alpha)}(x) =: \mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{\mathcal{N}}}\gamma^{5}_{\alpha\alpha}[\Sigma(x)+\tilde{\eta}(x)]} : \xi^{n}_{(\alpha)}(x) \tag{3.12}$$

$$\Pi^{n}_{(\alpha)}(x) =: \mathrm{e}^{-\mathrm{i}\sqrt{\frac{\pi}{\mathcal{N}}}\gamma^{5}_{\alpha\alpha}[\Sigma(x)+\tilde{\eta}(x)]} : \pi^{n}_{(\alpha)}(x). \tag{3.13}$$

The conserved vector current $\mathcal{J}_{\mu} \equiv \mathrm{J}_{\mu} + \ell_{\mu}$, is obtained by a gauge invariant point-splitting limit procedure[5] and is given by

$$\mathcal{J}_{\mu}(x) = -\sqrt{\frac{\mathcal{N}}{\pi}}\epsilon_{\mu\nu}\partial^{\mu}\Sigma + \ell_{\mu}(x) \tag{3.14}$$

[4] Note that the mass generated for the field Σ, and thus for $\mathcal{A}_{\mu}$, is the same as that generated for the gauge field in the generalized Schwinger model with $\mathcal{N}$ flavoured Fermi fields [18].

[5] See section 3.2.

where $\ell_\mu(x)$ is a longitudinal current,

$$\ell_\mu(x) = -\sqrt{\frac{\mathcal{N}}{\pi}}\partial_\mu(\eta(x) + \Phi(x)) \tag{3.15}$$

which generates zero-norm states from the vacuum,

$$\langle \ell_\mu(x)\ell_\nu(y)\rangle = 0 \qquad \forall(x, y). \tag{3.16}$$

3.1. Algebra of observables and physical Hilbert space

The higher-derivative local gauge theory is defined in terms of the set of field operators $\{\Psi^n, \Pi^n, \mathcal{A}_\mu\}$, $n = 0, 1, \ldots, \mathcal{N} - 1$, which provides the intrinsic mathematical description of the theory, and generates a local field algebra $\Im$. The indefinite inner product Hilbert space $\mathcal{H}$ of the local gauge formulation is given by

$$\mathcal{H} \equiv \Im|0\rangle. \tag{3.17}$$

It is worth remarking that, as stressed in [7–9], the bosonization technique used to obtain the operator solution of the model requires the use of a larger redundant Bose field algebra $\Im^B$, generated by the set of free fields $\{\Sigma, \tilde{\eta}, \phi_1, \ldots, \phi_{\mathcal{N}}\}$. These Bose fields are *building blocks* in terms of which the operator solution is constructed and *a priori* should not be considered as elements of the field algebra $\Im$ [7]. The field algebra $\Im$ is a proper subalgebra of $\Im^B$ and is recovered from a particular set of operators constructed from linear combinations and Wick-ordered exponentials of Bose fields. Not all functionals of the Bose fields belong to the field algebra $\Im$, nor do all vectors of $\mathcal{H}^B \doteq \Im^B|0\rangle$ belong to the state space $\mathcal{H}$, which is a proper subspace of $\mathcal{H}^B$.

Since Σ is a free massive field and $\Sigma \propto \varepsilon_{\mu\nu}\mathcal{F}^{\mu\nu} \in \Im$, the Hilbert space $\mathcal{H}$ can be factorized as a product [7, 8]

$$\mathcal{H} = \mathcal{H}_\Sigma \otimes \mathcal{H}'_{\tilde{\eta},\xi^0,\ldots,\xi^{\mathcal{N}-1},\pi^0,\ldots,\pi^{\mathcal{N}-1}} \tag{3.18}$$

where $\mathcal{H}_\Sigma$ is the Fock space of the free massive field Σ, $\mathcal{H}'$ is the closure of the space of states and is obtained by applying to the vacuum polynomials of all fields belonging to $\Im$ not containing Σ, i.e.,

$$\left\{: \mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{\mathcal{N}}}\gamma^5\tilde{\eta}(x)}\xi^n(x) :, : \mathrm{e}^{-\mathrm{i}\sqrt{\frac{\pi}{\mathcal{N}}}\gamma^5\tilde{\eta}(x)}\pi^n(x) :, J^\mu_{\mathrm{free}}(x), \ell^\mu(x)\right\} \tag{3.19}$$

with $n = 0, 1, \ldots, \mathcal{N} - 1$. The set of fields (3.19) defines a local field subalgebra $\Im' \subset \Im$, such that $\Im \equiv \Im(\Sigma, \Im')$, and the closure of the space of states is given by $\mathcal{H}' \doteq \Im'|0\rangle$.

The Hilbert space factorization (3.18) means that the general Wightman functions of the higher-derivative gauge theory factorize as

$$\mathcal{W} = \mathcal{W}_\Sigma \times \mathcal{W}' \tag{3.20}$$

where $\mathcal{W}_\Sigma$ represents the Wightman functions generated by the massive field Σ and $\mathcal{W}'$ represents the Wightman functions generated by the massless Bose fields $\{\tilde{\eta}, \phi^1, \ldots, \phi^{\mathcal{N}}\}$. Since the massless fields are independent, using decomposition (2.12), the Wightman functions $\mathcal{W}'$ factorize as

$$\mathcal{W}' = \mathcal{W}_{\tilde{\eta}} \times \mathcal{W}_{\phi^1} \times \cdots \times \mathcal{W}_{\phi^{\mathcal{N}}} = \mathcal{W}_{\tilde{\eta}} \times \mathcal{W}_\Phi \times \hat{\mathcal{W}}^f_{\varphi^1} \times \cdots \times \hat{\mathcal{W}}^f_{\varphi^{(\mathcal{N}-1)}} \tag{3.21}$$

where $\hat{\mathcal{W}}^f_{\varphi^j}$ carry the cluster-violating contributions arising from the space–time functions $f_j(x^\pm)$.

Following the standard procedure [17, 18], in order to obtain a physical interpretation of the theory we define the gauge invariant subspace $\widehat{\mathcal{H}}$, by requiring that

$$\widehat{\mathcal{H}} \doteq \{|\Psi\rangle \in \widehat{\mathcal{H}} \mid \ell_\mu^{(-)}|\Psi\rangle = 0\} \tag{3.22}$$

such that Gauss's law holds in a weak form in $\widehat{\mathcal{H}}$:

$$\langle\Phi|(\mathcal{J}^\nu(x) - \partial_\mu \mathcal{F}^{\mu\nu}(x))|\Psi\rangle = \langle\Phi|\ell^\nu(x)|\Psi\rangle = 0 \qquad \forall|\Phi\rangle,\ |\Psi\rangle \in \widehat{\mathcal{H}}. \tag{3.23}$$

The Hilbert space $\widehat{\mathcal{H}}$ is a representation of the field subalgebra $\widehat{\Im} \subset \Im$, $\widehat{\mathcal{H}} = \widehat{\Im}|0\rangle$, where

$$\widehat{\Im} = \widehat{\Im}(\Sigma, \widehat{\Im}') \tag{3.24}$$

is the gauge invariant field algebra generated from the net of subalgebras of the field Σ and of the proper subalgebra $\widehat{\Im}' \subset \Im'$, which commutes with ℓ_μ,

$$[\widehat{\Im}', \ell_\mu(x)] = 0. \tag{3.25}$$

Since the field combination $(\tilde{\eta} + \tilde{\Phi})$ commutes with the longitudinal current $\ell_\mu(x)$, we can define the field subalgebra $\widehat{\Im}' = \widehat{\Im}'\{f(\tilde{\eta} + \tilde{\Phi}), \varphi^1, \ldots, \varphi^{\mathcal{N}-1}\}$, and decompose $\widehat{\mathcal{H}}$ as a product

$$\widehat{\mathcal{H}} = \mathcal{H}_\Sigma \otimes \widehat{\mathcal{H}}'_{f(\tilde{\eta}+\tilde{\Phi}),\varphi^1,\ldots,\varphi^{\mathcal{N}-1}} \tag{3.26}$$

with

$$\widehat{\mathcal{H}}' \equiv \widehat{\Im}'|0\rangle \tag{3.27}$$

where $f(\tilde{\eta} + \tilde{\Phi})$ represents all polynomial functionals and derivatives of the field combination $(\tilde{\eta} + \tilde{\Phi})$ that commutes with $\ell_\mu(x)$.

In a higher-derivative local gauge theory, the violation of the cluster property arises for two distinct reasons. One of them is the well-known charge condensation mechanism that emerges in two-dimensional gauge theories [17–19], and is associated with the existence of infinitely delocalized condensed states which carry the free fermion $U(1)$ charge and chirality. These states are generated from the vacuum by Wick-ordered exponentials of the field combination $(\tilde{\eta} + \tilde{\Phi})$, which generate constant Wightman functions. The other mechanism that conspires on behalf of the cluster violation is the higher-derivative nature of the free theory, much as discussed in the preceding section. In this way, the asymptotic factorization property is violated for a class of states belonging to the Hilbert space $\widehat{\mathcal{H}}$. In order to circumvent this problem, and obtain a subspace of states satisfying a reasonable set of physically meaningful axioms, we can establish a kind of hierarchy of spaces as a prescription to isolate the physical (gauge invariant) subspace of states satisfying the asymptotic factorization property.

To begin with, consider the space of states $\mathcal{H} \doteq \Im(\Sigma, \Im')|0\rangle$. We define an element $\boldsymbol{\omega}$ of the field subalgebra $\widetilde{\Im}' \subset \Im'$, by the following condition in $\mathcal{H}$:

Let $\omega(x)$ be an element of the field subalgebra $\Im'$. Then, $\omega(x) \in \widetilde{\Im}'$ iff the following algebraic constraints are satisfied:

$$\begin{aligned} &\left\{\omega^*(x), :\mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5\tilde{\eta}(x)} : \xi^\ell(y)\right\} = 0 \qquad \forall(x,y) \quad \ell = N+1, \ldots, 2N \\ &\left\{\omega\ (x), :\mathrm{e}^{-\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5\tilde{\eta}(x)} : \pi^\jmath(y)\right\} = 0 \qquad \forall(x,y) \quad \jmath = 0, 1, \ldots, N-1. \end{aligned} \tag{3.28}$$

The field subalgebra $\widetilde{\Im}'$ is generated by

$$\left\{:\mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5\tilde{\eta}}\xi^N :, \ldots, :\mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5\tilde{\eta}}\xi^{2\mathcal{N}} :, :\mathrm{e}^{-\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5\tilde{\eta}}\pi^0 :, \ldots, :\mathrm{e}^{-\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5\tilde{\eta}}\pi^N :, J^\mu_{\text{free}}, \ell^\mu\right\}. \tag{3.29}$$

In the subspace $\widetilde{\mathcal{H}} = \widetilde{\mathfrak{I}}\{\Sigma, \widetilde{\mathfrak{I}}'\}|0\rangle = \mathcal{H}_\Sigma \otimes \widetilde{\mathcal{H}}'$, the field operator $\Psi^N(x)$ satisfies the first-order equation of motion in the weak form,

$$\mathcal{D}_\mp \Psi^N_{(\alpha)}(x) \approx 0 \tag{3.30}$$

that is,

$$\langle \omega | \mathcal{D}_\mp \Psi^N_{(\alpha)}(x) | \omega \rangle = 0 \qquad \forall |\omega\rangle = \omega |0\rangle \in \widetilde{\mathcal{H}}. \tag{3.31}$$

The subspace $\widetilde{\mathcal{H}}'$ contains zero-norm states such as

$$\begin{aligned} &: \mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5 \tilde{\eta}(x)} \xi^{N+1}(x) : |0\rangle, \ldots, : \mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5 \tilde{\eta}(x)} \xi^{2N}(x) : |0\rangle \\ &: \mathrm{e}^{-\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5 \tilde{\eta}(x)} \pi^{0}(x) : |0\rangle, \ldots, : \mathrm{e}^{-\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5 \tilde{\eta}(x)} \pi^{N-1}(x) : |0\rangle \end{aligned} \tag{3.32}$$

generated from the field subalgebra $\widetilde{\mathfrak{I}}'_o \subset \widetilde{\mathfrak{I}}'$. The states of $\widetilde{\mathcal{H}}$ can be accommodated as equivalence classes of

$$: \mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5 \tilde{\eta}(x)} \xi^{N}(x) : |0\rangle \tag{3.33}$$

and

$$: \mathrm{e}^{-\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5 \tilde{\eta}(x)} \pi^{N}(x) : |0\rangle \tag{3.34}$$

modulo

$$\left\{ : \mathrm{e}^{-\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5 \tilde{\eta}} \sum_{\kappa=1}^{N-l} (\partial_\mp)^{N+l} \check{\chi}^\kappa_-(y) : \right\} |0\rangle \equiv \widetilde{\mathcal{H}}'_o \qquad l = 1, 2, \ldots, N \tag{3.35}$$

where $\widetilde{\mathcal{H}}'_o$ is a null inner product subspace of $\widetilde{\mathcal{H}}'$, $\widetilde{\mathcal{H}}'_o = \widetilde{\mathfrak{I}}'_o|0\rangle$. Defining the quotient space

$$\widetilde{\mathcal{H}}_{N+1} \doteq \frac{\widetilde{\mathcal{H}}}{\widetilde{\mathcal{H}}'_o} \tag{3.36}$$

such that

$$\widetilde{\mathcal{H}}_{N+1} \doteq \widetilde{\mathfrak{I}}_{N+1}\{\Sigma, \widetilde{\mathfrak{I}}'_{N+1}\}|0\rangle \tag{3.37}$$

where the field subalgebra $\widetilde{\mathfrak{I}}'_{N+1}$ is generated by

$$\left\{ : \mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5 \tilde{\eta}(x)} \psi^{N+1}(x) :, : \mathrm{e}^{-\mathrm{i}\sqrt{\frac{\pi}{N}}\gamma^5 \tilde{\eta}(x)} \psi^{N+1^*}(x) :, J^\mu_{\mathrm{free}}(x), \ell^\mu(x) \right\}. \tag{3.38}$$

In the quotient space $\widetilde{\mathcal{H}}_{N+1}$ the equation of motion is satisfied in the strong form

$$\mathcal{D}_\mp \Psi^N(x) = 0. \tag{3.39}$$

The gauge invariant field subalgebra is given by $\widetilde{\mathfrak{I}}_{(gi)}\{\Sigma, \widetilde{\mathfrak{I}}''_{N+1}\} \subset \widetilde{\mathfrak{I}}_{N+1}\{\Sigma, \widetilde{\mathfrak{I}}'_{N+1}\}$, where $\widetilde{\mathfrak{I}}''_{N+1} \subset \widetilde{\mathfrak{I}}'_{N+1}$, and is defined by the subsidiary condition in $\widetilde{\mathcal{H}}_{N+1}$:

$$[\widetilde{\mathfrak{I}}''_{N+1}, \ell_\mu] = 0. \tag{3.40}$$

The semidefinite inner product gauge invariant subspace is defined by $\widetilde{\mathcal{H}}_{gi} \doteq \widetilde{\mathfrak{I}}_{(gi)}|0\rangle$.

Let $\widetilde{\mathfrak{I}}^o_{gi}$ be the field subalgebra of $\widetilde{\mathfrak{I}}_{gi}$ generated from the field combination $(\eta + \Phi)$ and which generates the zero-norm subspace $\widetilde{\mathcal{H}}^o_{gi}$. The quotient space $\widetilde{\mathcal{H}}^+_{gi}$ defined by

$$\widetilde{\mathcal{H}}^+_{gi} = \frac{\widetilde{\mathcal{H}}_{gi}}{\widetilde{\mathcal{H}}^o_{gi}} \tag{3.41}$$

is the positive definite inner product gauge invariant subspace.

3.2. Bilocals

In analogy with the bilocal operators introduced by Lowenstein–Swieca [17], a general bilocal operator of the higher-derivative theory, which is invariant under c-number and q-number free gauge transformations and thus belongs to the field algebra $\mathfrak{I}_{(gi)}$, can be formally defined in terms of the phase-space field variables as

$$B_{\alpha\alpha}(x, y) \simeq \sum_{n=0}^{\mathcal{N}-1} \Psi^n_{(\alpha)}(x) \left(\mathrm{e}^{\mathrm{i}e\int_x^y \mathcal{A}_\mu(z)\,\mathrm{d}z^\mu}\right) \Pi^n_{(\alpha)}(y) \in \mathfrak{I}_{(gi)}. \tag{3.42}$$

The bilocal operator (3.42) has the Lorentz transformation properties of a bilinear in 'spin' 1/2 fields. The symbol $\simeq$ indicates that the operator product still needs to be defined. Using (2.12) and (2.15), we can write the bilocal operator (3.42) in terms of normal-ordered Wick exponentials as

$$B_{\alpha\alpha}(x, y) = \mathcal{N}_{\alpha\alpha}(x-y) T_{\alpha\alpha}(x, y) \hat{F}_{\alpha\alpha}(x, y) \sigma_{(\alpha)}(x) \sigma^*_{(\alpha)}(y) \tag{3.43}$$

where $T_{\alpha\alpha}(x, y)$ is the bilocal operator which depends only on the field Σ,

$$T_{\alpha\alpha}(x, y) =: \mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{\mathcal{N}}}[\gamma^5_{\alpha\alpha}\Sigma(x) + \int_x^y \tilde{\partial}_\mu \Sigma(z)\,\mathrm{d}z^\mu - \gamma^5_{\alpha\alpha}\Sigma(y)]} : \tag{3.44}$$

and

$$\sigma_{(\alpha)}(x) =: \mathrm{e}^{\mathrm{i}\sqrt{\frac{\pi}{\mathcal{N}}}[\gamma^5_{\alpha\alpha}(\tilde{\eta}(x) + \tilde{\Phi}(x)) + (\eta(x) + \Phi(x))]} : \tag{3.45}$$

is a spurious operator which generates from the vacuum infinitely delocalized condensed states which carry the free fermion charge and chirality [15, 17, 18]. The $U(1)$-screened composed operator $\hat{F}_{\alpha\alpha}(x, y)$ depends on the $\mathcal{N}-1$ fields φ^{i_D}. The factor $\mathcal{N}_{\alpha\alpha}(x-y)$ carries the canonical free-fermion field short-distance singularity,

$$\mathcal{N}_{\alpha\alpha}(x-y) = \frac{\mathrm{i}}{2\pi(x-y)^{\pm}} \tag{3.46}$$

and ensures the correct Lorentz transformation properties of 'spin' 1/2 field bilocal for the bosonized composite operator (3.43).

The vector current (3.14) can be computed from the bilocal operator in (3.42) through the point-splitting limit procedure:

$$\mathcal{J}_{\pm}(x) = \lim_{\varepsilon\to 0}\{B_{\alpha\alpha}(x+\varepsilon, x) - V.E.V.\}. \tag{3.47}$$

The 'composite' spurious (neutral) dipole operator $\sigma^*_{(\alpha)}(x)\sigma_{(\alpha)}(y)$ is defined as an element of $\mathcal{H}$ and leads to constant vacuum expectation value

$$\langle 0|\sigma^*_{(\alpha)}(x)\sigma_{(\alpha)}(y)|0\rangle = 1. \tag{3.48}$$

The state $|\sigma^*_{(\alpha)}\sigma_{(\alpha)}\rangle$ is translationally invariant in $\widetilde{\mathcal{H}}_{gi}$. The position independence of this state can be seen by computing the general Wightman functions involving the operator $\sigma^*_{(\alpha)}(x)\sigma_{(\alpha)}(y)$ and all operators belonging to the local field algebra $\mathfrak{I}_{(gi)}$. Thus, for any operator $\mathcal{O}(h_z) = \int \mathcal{O}(z)h(z)\,\mathrm{d}^2z \in \mathfrak{I}_{(gi)}$ of polynomials in the smeared fields belonging to $\mathfrak{I}_{(gi)}$, the position independence of the operator $\sigma^*_{(\alpha)}\sigma_{(\alpha)}$ can be expressed in the weak form as

$$\begin{aligned}&\langle 0|\sigma^*_{(\alpha)}(x)\sigma_{(\alpha)}(y)\mathcal{O}\left(h_{z_1},\ldots,h_{z_n}\right)|0\rangle \\ &\qquad = \mathcal{W}(z_1,\ldots,z_n) \equiv \langle 0|\mathcal{O}\left(h_{z_1},\ldots,h_{z_n}\right)|0\rangle \qquad \forall \mathcal{O}(h) \in \mathfrak{I}_{(gi)}\end{aligned} \tag{3.49}$$

where $\mathcal{W}(z_1,\ldots,z_n)$ is a function independent of the space–time coordinates (x, y). The spurious operator $\sigma^*_{(\alpha)}\sigma_{(\alpha)}$ does not carry any charge selection rule, and since it commutes with all operators belonging to the field algebra $\mathfrak{I}_{(gi)}$, it is reduced to the identity operator in $\widetilde{\mathcal{H}}_{gi}$.

A bilocal operator belonging to the field subalgebra $\widetilde{\mathfrak{I}}_{(gi)}$ can also be introduced as

$$\widetilde{B}_{\alpha\alpha}(x, y) \simeq \Psi^N_{(\alpha)}(x) \left(\mathrm{e}^{\mathrm{i}e\int_x^y \mathcal{A}_\mu(z)\,\mathrm{d}z^\mu}\right) \Pi^N_{(\alpha)}(y) \in \widetilde{\mathfrak{I}}_{(gi)}. \tag{3.50}$$

3.3. Expectation value of observables

The net of local (gauge invariant) observables $\{\mathcal{O}\}$ is generated by the field-strength tensor $\mathcal{F}_{\mu\nu} = e\sqrt{\pi/\mathcal{N}}\epsilon_{\mu\nu}\Sigma$, the current $\mathcal{J}^{\mu}$ and the bilocal quantities $B(x-y)$. The Wightman functions of the bilocal operator $B(z_i - z_j)$ can be factorized in terms of Wightman functions of Wick-ordered exponentials of the massive free field Σ, arising from the operators $T(z_i - z_j)$ in (3.43), and Wightman functions of Wick exponentials of the $\mathcal{N}-1$ massless scalar fields φ_{i_D}, arising from the operators $\hat{F}(z_i - z_j)$ in (3.43). Nevertheless, the expectation values of observables, corresponding to the Σ-sector of the Hilbert space, can be obtained by considering expectation values with respect to normalized states. Every local gauge invariant field operator $\boldsymbol{\Omega} \in \{\mathcal{O}\}$, can be factorized as a product $\boldsymbol{\Omega} = \boldsymbol{\Theta}[\Sigma]G$, with $\boldsymbol{\Theta}[\Sigma] \in \Im_{\Sigma}$, $G \in \tilde{\Im}''$, and $\Im_{\Sigma}$ is the field algebra generated by the massive free field $\Sigma(x)$. The expectation value of an observable $\boldsymbol{\mathcal{A}} \in \Im_{\Sigma}$, in a normalized state constructed from the smeared operator $\boldsymbol{\Omega}(f)$, is given by

$$\langle\boldsymbol{\Omega}|\boldsymbol{\mathcal{A}}|\boldsymbol{\Omega}\rangle \doteq \frac{\langle 0|\boldsymbol{\Omega}^*(f)\boldsymbol{\mathcal{A}}\boldsymbol{\Omega}(f)|0\rangle}{\langle 0|\boldsymbol{\Omega}^*(f)\boldsymbol{\Omega}(f)|0\rangle} \equiv \frac{\langle 0|\boldsymbol{\Theta}^*(f)\boldsymbol{\mathcal{A}}\boldsymbol{\Theta}(f)|0\rangle}{\langle 0|\boldsymbol{\Theta}^*(f)\boldsymbol{\Theta}(f)|0\rangle} \doteq \langle\boldsymbol{\Theta}|\boldsymbol{\mathcal{A}}|\boldsymbol{\Theta}\rangle \tag{3.51}$$

where $\boldsymbol{\Omega}|0\rangle \in \widetilde{\mathcal{H}}$, and $\boldsymbol{\Theta}|0\rangle \in \mathcal{H}_{\Sigma} \doteq \Im_{\Sigma}|0\rangle$. As a matter of fact, in virtue of (3.8) and (3.51), the Hilbert subspace $\mathcal{H}_{\Sigma}$ is isomorphic to the corresponding massive sector of the generalized QED_2 with $\mathcal{N}$ flavoured Fermi fields [18].

In particular, we can consider the charge distribution of a dipole by computing the expectation value of the charge distribution $\mathcal{J}^0(z) = \sqrt{\mathcal{N}/\pi}\,\partial_{z^1}\tilde{\Sigma}(z)$, with respect to a normalized dipole state $B(x, y)|0\rangle$ [22, 23]. As a consequence of (3.8) and (3.51), and as occurs in the usual massless Schwinger model, the local charge of the electric poles at the end of string oscillates periodically in time with frequency $M = \sqrt{\mathcal{N}\pi}$. Except for the increasing of the oscillating frequency by a factor $\sqrt{\mathcal{N}}$, the physical situation is the same as that of the standard model [22, 23] and identical to that of the generalized Schwinger model with $\mathcal{N}$ flavoured fermions [18], contrary to the conjectures made in [12]. The total charge at the end of strings is unstable and oscillates periodically in time no matter what the separation between the charges.

4. Conclusions

We have discussed at length some structural aspects of the field algebra of a class of higher-derivative field theories, which were not appreciated and clarified in the preceding literature. In order to give a consistent explanation of the mathematical structure of the model, as well as to display the effect of the higher derivatives in the physical content of the model, we considered the construction of the Hilbert space based on the intrinsic field algebra generated by the fields and their space–time derivatives. Within this approach we obtained a general selection criterion for the proper field subalgebra that generates Wightman functions satisfying the asymptotic factorization property. The physical proper subspace of the Hilbert space constructed within this approach, modulo zero-norm states, turned out to be isomorphic to the Hilbert space of the model without higher derivative for any odd derivative number model. The even derivative number models turned out to be entirely devoid of physical content. For higher-derivative theories with local gauge invariance, besides the Lowenstein–Swieca condition, an additional condition must be imposed on the field algebra in order to obtain a physical subspace of states. In the physical subspace the observable content generated by the field-strength tensor $\mathcal{F}_{\mu\nu} = e\sqrt{\pi/\mathcal{N}}\Sigma$, the vector current $\mathcal{J}_{\mu} = -\sqrt{\mathcal{N}/\pi}\varepsilon_{\mu\nu}\Sigma$ and the bilocal $B(x, y)$, are isomorphic to those of the QED_2 with $\mathcal{N}$ flavoured Fermi fields discussed in [18].

Since the use of higher-derivative models has been the subject of great attention recently in the context of non-Abelian field quantization [21], a foundational investigation of the basic structural properties may offer a valuable lesson for the understanding of the underlying physical properties of the higher-derivative models in 3 + 1 dimensions. The present discussion points to the severe limitation in the use of higher-derivative field theories while keeping a precise set of QFT axioms.

Appendix A

For $\mathcal{N}$ even the functions $f_j(x^{\mp})$ are given by

$$f_j(x^{\mp}) = \frac{m^{1/2-N}}{\sqrt{2}}\left[\frac{(-\mathrm{i}mx^{\mp})^{j-1}}{(j-1)!} + \frac{(-\mathrm{i}mx^{\mp})^{2N-j}}{(2N-j)!}\right] \tag{1.1}$$

$$f_{j+N}(x^{\mp}) = (-1)^{j+1} f_j^*(x^{\mp}) \tag{1.2}$$

where $j \leqslant N$. For $\mathcal{N}$ odd,

$$f_j(x^{\mp}) = \frac{m^{1/2-N}}{\sqrt{2}}\left[\frac{(mx^{\mp}/2)^{j-1}}{(j-1)!} + \frac{(-mx^{\mp})^{2N+1-j}}{(2N+1-j)!}\right] \tag{1.3}$$

$$f_{2N+2-j}(x^{\mp}) = \frac{m^{1/2-N}}{\sqrt{2}}\left[\frac{(mx^{\mp}/2)^{j-1}}{(j-1)!} - \frac{(-mx^{\mp})^{2N+1-j}}{(2N+1-j)!}\right] \tag{1.4}$$

for $1 \leqslant j \leqslant N$, whereas

$$f_{N+1}(x^{\mp}) = \frac{(x^{\mp}/2)^N}{N!}. \tag{1.5}$$

Appendix B

$$\begin{aligned}&\mathcal{N} = 2\\ &\phi_1 = \frac{1}{\sqrt{2}}(\Phi + \varphi)\end{aligned} \tag{2.1}$$

$$\phi_2 = \frac{1}{\sqrt{2}}(\Phi - \varphi) \tag{2.2}$$

$$\begin{aligned}\hat{F}_{\alpha\alpha}(x, y) = {}& \frac{1}{\pi}\frac{1}{(x-y)^{\pm}} : \cos\{\sqrt{2\pi}(\varphi(x^{\pm}) - \varphi(y^{\pm}))\} :\\ &+ \frac{m}{2\pi}\frac{(x-y)^{\mp}}{(x-y)^{\pm}} : \sin\{\sqrt{2\pi}(\varphi(x^{\pm}) + \varphi(y^{\pm}))\} :\\ &- \mathrm{i}\frac{\mu m}{2\pi}(x-y)^{\mp} : \sin\{\sqrt{2\pi}(\varphi(x^{\pm}) + \varphi(y^{\pm}))\} : .\end{aligned} \tag{2.3}$$

The bilocal of the higher-derivative free theory is given by

$$d_{\alpha\alpha}(x, y) =: \mathrm{e}^{\mathrm{i}\sqrt{2\pi}(\Phi(x^{\pm}) - \Phi(y^{\pm}))} : \hat{F}_{\alpha\alpha}(x, y). \tag{2.4}$$

References

[1] Abdalla E, Abdalla M C and Rothe K D 1991 *Non-Perturbative Methods in 2 Dimensional Quantum Field Theory* (Singapore: World Scientific), and references quoted therein

[2] Fradkim E 1991 Field theories of condensed matter physics *Frontiers in Physics* (Reading, MA: Addison-Wesley), and references quoted therein

[3] Glöckle W, Nogami Y and Fukui I 1987 *Phys. Rev.* D **35** 584
Munakata Y, Sakamoto J, Ino T, Nakamae T and Yamamoto F 1990 *Prog. Theor. Phys.* **83** 84
Munakata Y, Sakamoto J, Ino T, Nakamae T and Yamamoto F 1990 *Prog. Theor. Phys.* **83** 835
Sakamoto J 1993 *Prog. Theor. Phys.* **89** 119
Sakamoto J and Heike Y 1998 Bosonic structure of a 2-dimensional fermion model with interaction among different species *Preprint* hep-th/9807073

[4] Strocchi F 1993 *Selected Topics on the General Properties of Quantum Field Theory (Lecture Notes on Physics vol 15)* (Singapore: World Scientific)

[5] Fröhlich J 1992 *Non-Perturbative Quantum Field Theory (Advanced Series in Mathematical Physics vol 15)* (Singapore: World Scientific)

[6] Schroer B 1997 *Ann. Phys., NY* (*Preprint* CBPF-NF-026/97)

[7] Morchio G, Pierotti D and Strocchi F 1988 *Ann. Phys., NY* **188** 217
Strocchi F 1993 *Selected Topics on the General Properties of Quantum Field Theory (Lecture Notes in Physics vol 51)* (Singapore: World Scientific)

[8] Carvalhaes C G, Belvedere L V, Boschi Filho H and Natividade C P 1997 *Ann. Phys., NY* **258** 210

[9] Carvalhaes C G, Belvedere L V, do Amaral R L P G and Lemos N A 1998 *Ann. Phys., NY* **269** 1

[10] Belvedere L V, Amaral R L P G, Rothe K D and Sholtz G 1998 *Ann. Phys., NY* **262** 132

[11] Halpern M B 1975 *Phys. Rev.* D **12** 1684
Halpern M B 1976 *Phys. Rev.* D **13** 337

[12] Barcelos-Neto J and Natividade C P 1991 *Z. Phys.* C **49** 511

[13] do Amaral R L P G, Belvedere L V, Lemos N A and Natividade C P 1993 *Phys. Rev.* D **47** 3443

[14] Belvedere L V 1993 *Braz. J. Phys.* **23** 304

[15] Belvedere L V, do Amaral R L P G and Lemos N A 1995 *Z. Phys.* C **66** 613

[16] Belvedere L V, do Amaral R L P G, Carvalhaes C G and Lemos N A 2000 *Int. J. Mod. Phys.* A 2237

[17] Lowenstein S and Swieca J A 1971 *Ann. Phys., NY* **68** 172

[18] Belvedere L V, Swieca J A, Rothe K D and Schroer B 1979 *Nucl. Phys.* B **153** 112

[19] Belvedere L V 1986 *Nucl. Phys.* B **276** 197

[20] Gonzáles-Arroyo A and Korthals Altes C P 1983 *Phys. Lett.* B **131** 396
Connes A and Rieffel M 1987 *Contemp. Math. Oper. Alg. Math. Phys.* **62** AMS 237
Doplicher S, Fredenhagen K and Roberts J E 1995 *Commun. Math. Phys.* **172** 187

[21] Gonzáles Arroyo A and Okawa M 1983 *Phys. Lett.* B **120** 174
Gonzáles Arroyo A and Okawa M 1983 *Phys. Rev.* D **27** 2397
Filk T 1996 *Phys. Lett.* B **376** 53
Amorim R and Barcelos-Neto J 2001 *J. Phys. A: Math. Gen.* **34** 8851

[22] Swieca J A 1977 *Fortschr. Phys.* **25** 303

[23] Rothe K D, Rothe H J and Stamatescu I O 1977 *Ann. Phys., NY* **105** 63